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LETTER TO THE EDITOR

Off-diagonal matrix elements in the semiclassical limit

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Abstract. We give a very simple proof of the convergence of the non-diagonal Wigner functions of the harmonic oscillator eigenstates $W_{n,n+k}(A,\phi)$ to $\delta(A-A_0)\mathrm{e}^{\mathrm{i}k\phi}$ in the semiclassical limit $n\to\infty$, $\hbar\to 0$ and $n\hbar = A$.

It is well known and proved in great generality [C] that for a completely integrable system, the eigenfunctions concentrate on the classical torus at high energy. This means that the diagonal matrix elements of the Weyl quantized observable F^W tend to the mean value of the classical observable F on the torus $A = A_0$ when $\hbar\to 0$ and $n\hbar\to A_0$:

$$\langle n|F^W|n\rangle \longrightarrow \int_{A=A_0} F(A,\phi)\,\mathrm{d}\phi.$$

In view of their definition, this is equivalent to the fact that the Wigner functions $W_{n,n}(A,\phi)$ tend to the distribution $\delta(A-A_0)$. There is also a 'folk theorem' stating that the non-diagonal matrix elements tend to the Fourier coefficients of F:

$$\langle n|F^W|n+k\rangle \longrightarrow \int_{A=A_0} F(A,\phi)\mathrm{e}^{\mathrm{i}k\phi}\,\mathrm{d}\phi \tag{1}$$

Equivalently

$$W_{n,n+k}(A,\phi) \longrightarrow \delta(A-A_0)\mathrm{e}^{\mathrm{i}k\phi}.$$

In the case of the harmonic oscillator, this can be verified easily when F is a polynomial (see for instance [BV] for a short proof). Recently, Ripamonti proved this result for smooth observables [R]. Her proof is based on the explicit expression of the $W_{n,n+k}$ computed in the Bargman representation. Since the $W_{n,n+k}$ are Laguerre polynomials of order n, the limit $n\to\infty$ involves a long and difficult calculation using the asymptotic properties of the Laguerre polynomials. We propose in this letter a very simple proof of the same result. Our proof uses the properties of the anti-normal quantization, which fits better with the Bargman representation, to prove eventually the result (1) for Weyl quantization.

Let

$$H(q,p) = \sum_{i=1}^{d} \frac{1}{2}(p_i^2 + \omega_i^2 q_i^2)$$

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be the classical Hamiltonian of the d-dimensional harmonic oscillator on the phase space $\mathbb{R}^{2d} = \mathrm{T}^*\mathbb{R}^d$. We identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via

$$z_i = \frac{\omega_i q_i + \mathrm{i} p_i}{\sqrt{2\omega_i}}.$$

The action and angle variables are defined by $z_i = \sqrt{A_i}\mathrm{e}^{\mathrm{i}\phi_i}$ or $z = \sqrt{A}\mathrm{e}^{\mathrm{i}\phi}$ for short. The energy now reads

$$E = \sum_{i=1}^{d} \omega_i A_i = \omega \cdot A. \tag{2}$$

The classical trajectories in phase space are the poly-circles $A =$ constant. For any classical observable $F(z) = F(A, \phi)$ we denote by $\langle F \rangle_A$ the mean value of F on the classical trajectory $|z_i| = \sqrt{A_i}$:

$$\langle F \rangle_A = \frac{1}{(2\pi)^d} \int F(A_1, \ldots, A_d, \phi_1, \ldots, \phi_d)\, \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_d = \frac{1}{(2\pi)^d} \int F(A, \phi)\, \mathrm{d}\phi.$$

The Hilbert space of quantum states can be realized on $\mathbb{C}^d$ as a space of entire functions:

$$\mathcal{B} = \left\{ f;\; f \text{ entire on } \mathbb{C}^d,\; \int |f(z)|^2 \mathrm{e}^{-|z|^2/\hbar} \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{(\pi\hbar)^d} < \infty \right\}$$

where $z = (z_1, \ldots, z_d)$, $\mathrm{d}z\, \mathrm{d}\bar{z}$ is the Lebesgue measure on $\mathbb{C}^d$, and $|z|^2 = \sum_{i=1}^{d} |z_i|^2$. On this space the quantized Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{d} \omega_i \left(\hbar z_i \frac{\partial}{\partial z_i} + \frac{\hbar}{2} \right).$$

The normalized steady states are given by

$$\langle z | n \rangle = \frac{z^n}{\sqrt{n!}\hbar^{|n|/2}}$$

where here and hereafter $n = (n_1, \ldots, n_d)$ is a multi-index and $z^n = z_1^{n_1} \cdots z_d^{n_d}$, $n! = n_1! \cdots n_d!$ and $|n| = \sum_{i=1}^{d} n_i$. The energy of this state is given by

$$\begin{aligned} E_n &= \langle n | \mathcal{H} | n \rangle \\ &= \sum_{i=1}^{d} (n_i + \tfrac{1}{2}) \hbar \omega_i \\ &= \hbar \left(n + \left\{ \tfrac{1}{2} \right\} \right) \cdot \omega \end{aligned} \tag{3}$$

where $\left\{ \frac{1}{2} \right\}$ is the multi-index defined by $\left\{ \frac{1}{2} \right\} = \left(\frac{1}{2}, \ldots, \frac{1}{2} \right)$.

The Weyl–Wigner correspondence allows construction of a quantum observable F^W corresponding to any smooth classical observable F (say in $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$). We now state the main result.

Theorem 1. Let F and F^W be as above. Let n and k be multi-indices. Suppose that $n \to \infty$ and $\hbar \to 0$ in such a way that $(n + \{\frac{1}{2}\})\hbar \to A$ (see equations (2) and (3)), then

$$\langle n | F^W | n + k \rangle \to \int_{A = A_0} F(A, \phi) \mathrm{e}^{\mathrm{i}k \cdot \phi} \frac{\mathrm{d}\phi}{(2\pi)^d}$$

or equivalently

$$W_{n, n+k}(A, \phi) \to \delta(A - A_0) \mathrm{e}^{\mathrm{i}k \cdot \phi}.$$

Proof. We recall first the definition of the anti-normal (or anti-Wick) quantization. Let F be a classical observable as above. We define the anti-normal quantum observable F^{AN} as follows:

$$\left(F^{AN}\psi\right)(z) = \int \mathrm{e}^{\bar{w}\cdot z/\hbar} F(w)\psi(w)\mathrm{e}^{-|w|^2/\hbar} \frac{\mathrm{d}w\,\mathrm{d}\bar{w}}{(\pi\hbar)^d}$$

for any $\psi \in \mathcal{B}$. It is well known that the anti-normal and Weyl quantizations are equivalent in the limit $\hbar \to 0$. More precisely, one has [HMR] that for any classical observable as above $F^W - F^{AN} \to 0$ in the operator norm sense when $\hbar \to 0$: $\|F^W - F^{AN}\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \to 0$. Since the states $|n\rangle$ are normalized, it follows from this that

$$\lim\langle n|F^W|n+k\rangle = \lim\langle n|F^{AN}|n+k\rangle.$$

As a result, we just have to calculate the limit on the right-hand-side. This turns out to be much easier than dealing directly with the left-hand-side, as is done in [R]. A standard calculation now gives:

$$\begin{aligned}\langle n|F^{AN}|n+k\rangle &= \int \langle n|z\rangle\langle z|F^{AN}|n+k\rangle \mathrm{e}^{-|z|^2/\hbar} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(\pi\hbar)^d} \\ &= \int \frac{\bar{w}^n}{\sqrt{n!}\,\hbar^{|n|/2}} F(w) \frac{w^{n+k}}{\sqrt{n+k!}\,\hbar^{|n+k|/2}} \mathrm{e}^{-|w|^2/\hbar} \frac{\mathrm{d}w\,\mathrm{d}\bar{w}}{(\pi\hbar)^d}.\end{aligned}$$

We transform this integral using action-angle coordinates: more precisely, let (v, ϕ) be new variables defined by $w = \sqrt{Av}\mathrm{e}^{\mathrm{i}\phi}$, that is to say $w_i = \sqrt{A_i v_i}\mathrm{e}^{\mathrm{i}\phi_i}$ for $i = 1, \ldots, d$. Then

$$\begin{aligned}\langle n|F^{AN}|n+k\rangle &= \int \frac{\left(\sqrt{vA}\mathrm{e}^{-\mathrm{i}\phi}\right)^n}{\sqrt{n!}\,\hbar^{|n|/2}} F(vA,\phi) \frac{\left(\sqrt{vA}\mathrm{e}^{\mathrm{i}\phi}\right)^{n+k}}{\sqrt{(n+k)!}\,\hbar^{|n+k|/2}} \mathrm{e}^{-A\cdot v/\hbar} A^{\{1\}} \frac{\mathrm{d}v\,\mathrm{d}\phi}{(2\pi\hbar)^d} \\ &= \int \frac{v^{n+k/2} A^{n+k/2+\{1\}} \quad \mathrm{e}^{\mathrm{i}k\cdot\phi} F(vA,\phi)\mathrm{e}^{-A\cdot v/\hbar}}{\sqrt{n!}\sqrt{(n+k)!}\,\hbar^{|n|+|k|/2+d}} \frac{\mathrm{d}v\,\mathrm{d}\phi}{(2\pi)^d}\end{aligned}$$

where $\{1\}$ is the multi-index $(1, \ldots, 1)$. Since we have $(n + \{\frac{1}{2}\})\hbar \to A$, we can write $A = na_n\hbar$, i.e. $A_i = a_{n_i} n_i \hbar$ where $a_n = (a_{n_1}, \ldots, a_{n_d}) \to (1, \ldots, 1)$ when $n \to \infty$, we obtain:

$$\langle n|F^{AN}|n+k\rangle = \int v^n \frac{(na_n)^{n+k/2+\{1\}}}{\sqrt{n!}\sqrt{(n+k)!}} \mathrm{e}^{-(na_n)\cdot v}\, \mathrm{e}^{\mathrm{i}k\cdot\phi} F(vA,\phi) v^{k/2} \frac{\mathrm{d}v\,\mathrm{d}\phi}{(2\pi)^d}.$$

In order to conclude we just have to prove, in the sense of distributions, that

$$v^n \frac{(na_n)^{n+k/2+\{1\}}}{\sqrt{n!}\sqrt{(n+k)!}} \mathrm{e}^{-(na_n)\cdot v} \overset{n\to\infty}{\longrightarrow} \delta(v - \{1\})$$

that is to say, coordinate by coordinate:

$$v_i^{n_i} \frac{(n_i a_{n_i})^{n_i+k/2+1}}{\sqrt{n_i!}\sqrt{(n_i+k_i)!}} \mathrm{e}^{-(n_i a_{n_i})v_i} \overset{n_i\to\infty}{\longrightarrow} \delta(v_i - 1).$$

This follows from an elementary calculation. Applying this result we obtain

$$\langle n|F^{AN}|n+k\rangle \to \int \mathrm{e}^{\mathrm{i}k\cdot\phi} F(vA,\phi) \frac{\mathrm{d}\phi}{(2\pi)^d}$$

which proves the theorem. □

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